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The Spin Statistics Connection for Dyons

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Goldhaber's spin-statistics connection for electric-pole–magnetic-pole composite "dyons" is deduced in a gauge-invariant way by connecting the spatial interchange operator with a rotation.

It has been known for some time, and has recently been stressed in the context of non-Abelian gauge theories, that wave functions representing spinless particles can nevertheless transform under rotations as spinors, in a theory involving monopoles. Furthermore, Goldhaber (1976)[3] has recently shown in the context of nonrelativistic quantum mechanics that two interacting electric-pole(e)–magnetic-pole(g) composite systems (which we will henceforth refer to as "dyons") behave as fermions when interchanged. In this comment we further discuss this result. Our purpose is threefold: (i) by considering the relevant rotation operators (Zumino, 1966; Frenkel and Hrasko, 1975; Bais et al.), we recall how dyons transform under rotations; (ii) by treating the interchange operation (in a special case) as a rotation by π about the center-of-mass of the two-dyon system, we show how the dyons behave upon interchange; and (iii) by defining a gauge-invariant interchange operator, we explicitly establish the gauge invariance of the spin-statistics connection for dyons. It is clearly seen in this way that the spinor and fermionic behaviors of the dyons have exactly the *same* origin, namely, the

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need to rotate Dirac strings by gauge transformations in order to achieve manifest rotational invariance. The spin-statistics connection is thus confirmed in this interesting quantum-mechanical system.

For clarity we will first outline our arguments and afterwards supply technical details. Consider first a composite "dyon" consisting of an electrically charged particle moving in the magnetic field $\mathbf{H} = \mu\hat{r}/r^2$ of a fixed monopole with magnetic charge $g \equiv 4\pi\mu$. The Schrödinger equation for the charged-particle wave function involves the vector potential

$$\mathbf{A}_\mathbf{n}(\mathbf{r}) = \frac{1}{r}\frac{\hat{r} \times \hat{n}}{1 - \hat{r}\cdot\hat{n}} \tag{1}$$

whose curl differs from $\mathbf{H}/\mu$ by a string singularity from the monopole to infinity along the direction $\hat{n}$. Provided the Dirac quantization condition (Dirac, 1931)

$$\epsilon \equiv e\mu \equiv \frac{eg}{4\pi} = \frac{n}{2} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{2}$$

is satisfied, the string can be arbitrarily moved by an acceptable gauge transformation and is, therefore, unobservable (Brandt and Primack, 1977).

The conserved angular momentum operator in the above theory is

$$\mathbf{J}_\mathbf{n} = \mathbf{r} \times (-i\nabla - \epsilon\mathbf{A}_\mathbf{n}) - \epsilon\hat{r} \equiv \mathbf{l} + \mathbf{j}_\mathbf{n} \tag{3}$$

where

$$\mathbf{l} = \mathbf{r} \times (-i\nabla) \tag{4}$$

generates coordinate rotations $\mathbf{r} \to e^{-i\omega\cdot\mathbf{l}}\mathbf{r}$ and

$$\mathbf{j}_\mathbf{n} = -\epsilon(\mathbf{r} \times \mathbf{A}_\mathbf{n} + \hat{r}) \tag{5}$$

generates the corresponding gauge transformation which preserves the original direction of the string. Thus the transformation $\psi_\mathbf{n}(\mathbf{r}) \to \mathscr{T}_\mathbf{n}(\boldsymbol{\omega})\psi_\mathbf{n}(\mathbf{r})$ generated by (3) is a transformation from one solution of the Schrödinger equation to another solution constructed using the vector potential (1). Infinitesimal rotations about $\hat{n}$ are generated by

$$\hat{n}\cdot\mathbf{J}_\mathbf{n} = \hat{n}\cdot\mathbf{l} + \epsilon \tag{6}$$

which can be immediately integrated to give the finite rotation operator

$$\mathscr{T}_\mathbf{n}(\omega\hat{n}) = e^{-i\omega n\cdot\mathbf{l} - i\omega\epsilon} \tag{7}$$

In particular,

$$\mathscr{T}_\mathbf{n}(2\pi\hat{n}) = (-1)^{2\epsilon} \tag{8}$$

and so the composite dyon is seen to transform under rotations as spin ϵ. A more detailed analysis (Zumino, 1966; Frenkel and Hrasko, 1975; Bais et al.) confirms that

$$\mathscr{T}_{\mathbf{n}}(2\pi\hat{\omega}) = (-1)^{2\epsilon}$$

for 2π rotations about an arbitrary axis $\hat{\omega}$ as required by rotational invariance. Nothing in the above discussion is changed if a potential $V(r)$ is added to the Hamiltonian to provide (nonelectromagnetic) binding between e and g.

To investigate the behavior of the composite dyons considered above upon interchange, we consider next a system of two identical dyons $(e_1, g_1) = (e_2, g_2) = (e, g)$. The Schrödinger equation separates with respect to the overall center-of-mass $\mathbf{r}_1 + \mathbf{r}_2$ ($\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinates of the center-of-mass of the individual dyons), and so we need only concern ourselves here with the dependence of the wave function $\psi_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{r})$ on the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the dyon strings $\mathbf{n}_1$ and $\mathbf{n}_2$ [see equations (22) and (23)], and the internal coordinates of the dyon. The string directions can again be individually rotated arbitrarily by a gauge transformation, and it is convenient to choose identical strings $\hat{n}_1 = \hat{n}_2 = \hat{n}$, because then the Hamiltonian for the system is invariant under the interchange $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ of the dyon coordinates, and so $\psi_{\mathbf{n}_1\mathbf{n}_1}(-\mathbf{r})$ is then also a solution of the Schrödinger equation. It is only in this case that we can make contact with the usual requirements on the symmetry of wave functions, namely, that if the individual particles e and g are both bosons (which we shall assume for simplicity) then ψ must be symmetric under $\mathbf{r} \to -\mathbf{r}$:

$$\psi_{\mathbf{nn}}(\mathbf{r}) = \psi_{\mathbf{nn}}(-\mathbf{r}) \tag{9}$$

Ordinarily, $\mathbf{r} \to -\mathbf{r}$ actually represents the physical interchange of the two identical particles. We can see this by imagining the interchange to take place physically, for those values of the relative coordinate $\mathbf{r}$ lying in the plane through $\mathbf{r} = 0$ perpendicular to any given axis $\hat{n}$, by means of a 180° rotation about $\hat{n}$:

$$\mathscr{I}\psi(\mathbf{r}) = \mathscr{R}(\pi\hat{n})\psi(\mathbf{r}) = \psi(-\mathbf{r}) \qquad \mathbf{r}\cdot\hat{n} = 0 \tag{10}$$

where $\mathscr{I}$ is the interchange operator and $\mathscr{R}$ is the rotation operator

$$\mathscr{R}(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot\mathbf{l}} \tag{11}$$

where the orbital angular momentum (4) generates rotations of the relative coordinate $\mathbf{r}$. (Spin is here an inessential complication and we ignore it.) When there are monopoles interacting with electric charges, however, we have already seen that the rotation operator must be generalized to

$$\mathscr{T}_{\mathbf{n}}(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot\mathbf{J}_{\mathbf{n}}} \tag{12}$$

where $\mathbf{J}_{\mathbf{n}}$, equation (3), generates a gauge transformation together with a coordinate rotation. The corresponding generalization of (10) is

$$\mathscr{I}\psi_{\mathbf{nn}}(\mathbf{r}) = \mathscr{T}_{\mathbf{nn}}(\pi\mathbf{n})\psi_{\mathbf{nn}}(\mathbf{r}) \qquad \mathbf{r}\cdot\hat{n} = 0 \tag{13}$$

where

$$\mathscr{T}_{\mathbf{nn}}(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot\mathbf{J}_{\mathbf{nn}}} \tag{14}$$

and the operator which generates rotations of the relative coordinate is

$$\mathbf{J}_{\mathbf{n}_1\mathbf{n}_2} = \mathbf{r} \times \{-i\boldsymbol{\nabla}_r + \epsilon[\mathbf{A}_{\mathbf{n}_1}(\mathbf{r}) - \mathbf{A}_{\mathbf{n}_1}(-\mathbf{r})]\} \tag{15}$$

where $\mathbf{A}_{\mathbf{n}}$ is defined by (1) and

$$\epsilon \equiv e_1\mu_1 = e_2\mu_2$$

Note that the rotation operator (14) does not affect the internal coordinates of the dyons [see equation (23)], since the relative coordinate ($\mathbf{r}$) and momentum ($\boldsymbol{\pi}$) [see equation (24)] and the angular momentum operator (15) constructed from them all commute with the internal coordinates. It will therefore enable us to construct a bona fide interchange operator. Since

$$\hat{n}\cdot\mathbf{J}_{\mathbf{nn}} = \hat{n}\cdot\mathbf{l} + 2\epsilon \tag{16}$$

we find immediately that

$$\mathscr{T}_{\mathbf{nn}}(\pi\hat{n}) = (-1)^{2\epsilon}e^{-i\pi\hat{n}\cdot\mathbf{l}} \tag{17}$$

Hence

$$\mathscr{I}\psi_{\mathbf{nn}}(\mathbf{r}) = (-1)^{2\epsilon}\psi_{\mathbf{nn}}(-\mathbf{r}) = (-1)^{2\epsilon}\psi_{\mathbf{nn}}(\mathbf{r}) \tag{18}$$

Thus for $\epsilon = \frac{1}{2}, \frac{3}{2}, \ldots$, the dyons indeed behave as fermions under interchange even though they are made out of bosons. Comparison of equations (8) and (18) provides the usual connection between spin and statistics and exposes the identical origin of the $\epsilon = \frac{1}{2}$ dyon's half-integer spin and fermionic behavior under interchange.

We proceed to supply the details which establish the above conclusions. Consider first a single composite dyon system. The charged-particle wave functions $\psi_{\mathbf{n}}(\mathbf{r})$ which are solutions of the Schrödinger equation

$$\frac{1}{2m}(-i\boldsymbol{\nabla} - \epsilon\mathbf{A}_{\mathbf{n}})^2\psi_{\mathbf{n}} = E\psi_{\mathbf{n}} \tag{19}$$

are single valued because of the quantization condition (2). The string is not observable because its position can be arbitrarily changed by a gauge transformation. This theory is rotationally invariant because the effect of a rotation $\mathbf{r} \to R^{-1}(\boldsymbol{\omega})\mathbf{r}$ on a string can be undone by a gauge transformation. The actual rotation operators are given by (see Zumino, 1966; Frenkel and Hrasko, 1975; Bais et al.)

$$\mathscr{T}_{\mathbf{n}}(\boldsymbol{\omega}) = \exp i\epsilon[\Omega_{R(\boldsymbol{\omega})\mathbf{n},\mathbf{n}}(\mathbf{r}) - \eta_{\mathbf{n}}(\boldsymbol{\omega})]\mathscr{R}(\boldsymbol{\omega}) \tag{20}$$

where $\Omega_{\mathbf{n}',\mathbf{n}}(\mathbf{r})$ is the solid angle of the infinite surface bounded by $\mathbf{n}$ and $\mathbf{n}'$ as seen from $\mathbf{r}$, and

$$\eta_{\mathbf{n}}(\boldsymbol{\omega}) = \alpha_{\omega} + \gamma_{\omega} \tag{21}$$

in terms of the Euler angles $(\alpha_{\omega}, \beta_{\omega}, \gamma_{\omega})$ corresponding to the rotation $\boldsymbol{\omega}$. It is not difficult to check that the infinitesimal rotations given by (20) are generated by (3). In particular, the form (20) immediately gives equation (8), using (21), and also immediately gives $\mathscr{T}_{\mathbf{n}}(2\pi\hat{\omega}) = (-1)^{2\epsilon}$ for $\hat{\omega}\cdot\hat{n} = 0$, using the definition of $\Omega_{\mathbf{n}',\mathbf{n}}$ (an entire plane subtends a solid angle of $\pm 2\pi$ from any viewpoint).

We next take up the system of two identical dyons. Neglecting for simplicity the internal structure of the dyons, the Hamiltonian in relative coordinates is

$$\mathscr{H}_{\mathbf{n}_1\mathbf{n}_2} = \frac{\mathbf{P}^2}{4M} + \frac{(\mathbf{p} - \epsilon\mathbf{A}_{\mathbf{n}_1\mathbf{n}_2})^2}{M} + \frac{e^2 + g^2}{4\pi r} \tag{22}$$

where

$$\begin{aligned} \mathbf{P} &= \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) \\ \mathbf{p} &= \mathbf{p}_1 - \mathbf{p}_2 \\ \mathbf{A}_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{r}) &= \mathbf{A}_{\mathbf{n}_1}(\mathbf{r}) - \mathbf{A}_{\mathbf{n}_2}(-\mathbf{r}) \end{aligned} \tag{23}$$

and M is the mass of each dyon. Equation (22) leads to the correct nonrelativistic Heisenberg equation of motion, and the corresponding Schrödinger equation is invariant under independent gauge transformations on each vector potential. The kinetic momentum conjugate to $\mathbf{r}$ is (Goldhaber, 1976)

$$\boldsymbol{\pi} = im[\mathscr{H}, \mathbf{r}] = \mathbf{p} - \epsilon\mathbf{A}_{\mathbf{n}_1\mathbf{n}_2} \tag{24}$$

and so the generator of rotations of $\mathbf{r}$ is (15)

$$\mathbf{J} = \mathbf{r} \times \boldsymbol{\pi} \tag{25}$$

since the total electromagnetic angular momentum vanishes. It can indeed be immediately checked that (25) does generate the desired rotations by using the commutation relations[4]

$$[\pi^i, \pi^j] = 0 \qquad [r^i, r^j] = 0 \qquad [r^i, \pi^j] = i\delta^{ij} \tag{26}$$

[4] These are true as operator equations on the Hilbert space of eigenfunctions of $\mathscr{H}_{\mathbf{n}_1,\mathbf{n}_2}$ [see Brandt and Primack, 1977].

The Hamiltonian (22) is invariant under the interchange $r_1 \leftrightarrow r_2$ (i.e., $r \leftrightarrow -r$, $\mathbf{p} \leftrightarrow -\mathbf{p}$) if the strings are chosen to lie in the same direction, say $\hat{n}$.[5]

We have stressed that only in the gauge $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}$ can the equation (9), which expressed the bosonic nature of the constituents, be assumed. In an arbitrary gauge, this then implies that the constituents described by $\psi_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{r})$ are bosonic if this wave function, when transformed by a gauge transformation $\mathscr{G}$ to the $\hat{n}_1 = \hat{n}_2 = \hat{n}$ gauge, satisfies (9); i.e., if

$$\psi_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{r}) = \mathscr{G}_{\mathbf{n}_1\mathbf{n},\mathbf{n}_2\mathbf{n}}(\mathbf{r})\psi_{\mathbf{nn}}(\mathbf{r}) \tag{27}$$

then the definition

$$\mathscr{I}_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{r}) = \mathscr{G}_{\mathbf{n}_1\mathbf{n},\mathbf{n}_2\mathbf{n}}(\mathbf{r})\mathscr{I}\mathscr{G}^{-1}_{\mathbf{n}_1\mathbf{n},\mathbf{n}_2\mathbf{n}}(\mathbf{r}) \tag{28}$$

together with (18) gives

$$\mathscr{I}_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{r})\psi_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{r}) = (-1)^{2\epsilon}\psi_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{r}) \tag{29}$$

trivially establishing the gauge invariance of the spin-statistics connection for dyons.[6]

Our argument is analogous to that of Goldhaber (1976), but has the virtue of being manifestly gauge invariant. Goldhaber invokes the Bose symmetry under interchange of the dyon constituents in the $\hat{n}_1 = \hat{n}_2 = \hat{n}$ gauge (see his footnote 7) to conclude that his wave function $\Psi(\mathbf{r})$ ($= \psi_{\mathbf{nn}}$ in our notation) is symmetric under interchange. He then transforms to a gauge $\hat{n}_1 = -\hat{n}_2 = \hat{n}$ in which $\mathbf{A}_{\mathbf{n}_1\mathbf{n}_2}$ vanishes, points out that the corresponding wave function $\Phi = e^{-i\alpha}\Psi$ (see footnote 7 below) is antisymmetric under interchange, and concludes that the dyons are therefore fermions. Indeed, in that gauge, our interchange operator $\mathscr{I}_{\mathbf{n},-\mathbf{n}}$ has only the effect $\mathbf{r} \rightarrow -\mathbf{r}$ without an additional gauge transformation (since $\mathbf{J}_{\mathbf{n},-\mathbf{n}} = \mathbf{r} \times \mathbf{p}$) so the

[5] For the choice of strings $\hat{n}_1 = -\hat{n}_2 = \hat{n}$, $\mathbf{A}_{\mathbf{n}_1\mathbf{n}_2}$ vanishes (the existence of such a gauge is obvious since the curl of $\mathbf{A}_{\mathbf{n}_1\mathbf{n}_2}$ is zero) and the Hamiltonian (22) again becomes interchange symmetric. Our construction of $\mathscr{I}$ in this case would then *not* produce the $(-1)^{2\epsilon}$ factor. (We thank P. Hasenfratz for raising this question.) The point is that it is implicit in our entire discussion that our dyons are really charged boson–monopole–boson composites so that the Hamiltonian actually contains the additional term

$$\epsilon\mathbf{p}_1\cdot\left[\frac{\mathbf{A}_{\mathbf{n}_1}(\mathbf{r})}{m_e} + \frac{\mathbf{A}_{\mathbf{n}_2}(-\mathbf{r})}{m_g}\right] + \epsilon\mathbf{p}_2\cdot\left[\frac{\mathbf{A}_{\mathbf{n}_1}(-\mathbf{r})}{m_e} + \frac{\mathbf{A}_{\mathbf{n}_2}(\mathbf{r})}{m_g}\right]$$

involving the dyon internal momenta, and thus the Hamiltonian is exchange invariant *only* if $\hat{n}_1 = \hat{n}_2$. [However, if $\epsilon = 0, \pm 1, \pm 2, \ldots$, the term becomes exchange invariant also for $\hat{n}_1 = -\hat{n}_2 = \hat{n}$ if Schwinger's vector potential $\frac{1}{2}(A_\mathbf{n} + A_{-\mathbf{n}})$ is used, but then dyons never behave as fermions.]

[6] This shows in particular that the same conclusions are valid in the nonsingular formulation of monopole theory due to Wu and Yang, Phys. Rev. D **12**, 3845 (1975). The equivalence of this approach with that of Dirac is shown in Brandt and Primack (1976).

[7] This equation, with $\alpha(\mathbf{r}) = \epsilon \int^{\mathbf{r}} d\boldsymbol{\xi}\cdot[A_\mathbf{n}(\boldsymbol{\xi}) - \mathbf{A}_{-\mathbf{n}}(\boldsymbol{\xi})]$, is a special case of our relation (27) and $\Phi = \psi_{\mathbf{n},-\mathbf{n}}$ is our notation.

$(-1)^{2\epsilon}$ symmetry of the wave function under $\mathbf{r} \to -\mathbf{r}$ in this gauge *does* give the interchange eigenvalue.

In Goldhaber's approach, the only assumption needed was (9), whereas we required both (9) and (13). We could also proceed without invoking (13) by arguing (as does Goldhaber) that the interchange operator $\mathscr{I}$ *must* reduce simply to coordinate interchange in the $\mathbf{n}_1 = -\mathbf{n}_2$ gauge in which $\mathbf{A}_{\mathbf{n}_1\mathbf{n}_2} = 0$.

In conclusion, let us stress that neither we nor Goldhaber have *proved* that the usual spin-statistics connection is valid for dyons. Rather, we have shown that the assumption (9) directly implies this connection. This suggests that the connection *can* actually be proved if and when a complete quantum field theory of magnetic charge is constructed.

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